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A COMMUTATIVE-EXTREMAL EXTENSION OF HILBERT-SCHMIDT THEOREM

CONSTANTIN P. NICULESCU

The leading idea in writing this paper was that most mathematical objects are determined by their singularities in a way recalling more or less the Krein-Milman Theorem. That is merely a philosophical point of view until the mathematical concept of a singularity is precised. We shall be concerned here with the case where a singular point is meant as an extreme point (in a generalized sense).

In our paper the notion of an extreme point is associated to an order relation of a special character and the main problem which we are dealing with is to decompose the given Banach space E into smaller pieces geometrically determined, particularly to represent the elements of E as series of discrete elements. To different order relations correspond generally different notions of extremality and thus we are able to bring together notions such as of an extreme point (in the classical sense), atom (in Banach lattice theory) and eigenvector.

Our main result asserts that under reasonable restrictions if A is a self-adjoint compact operator acting on a Banach space E then $\overline{\mathrm{Im}\, A}$ is a complemented subspace having an unconditional finite dimensional decomposition (in the sense of [LT 1]). See Theorem 5.1 and Proposition 5.4 below.

Section 1 contains preliminaries on Alfsen-Effros type order relations. The first two examples go back to [AE] but the systematic study was started around the 83's by the author. See [N1]—[N3].

In section 2 we introduce the notion of a facial cone (associated to an Alfsen-Effros type order relation) and outline an analogue of the principal ideal theory.

Section 3, devoted to the concept of an extreme point (associated to an Alfsen-Effros type order relation) contains a complete description of the finite dimensional case. See Theorem 3.9 below. That theorem brings together classical results due to Cauchy, Caratheodory and Yudin.

In section 4 we show how to associate to any order relation of Alfsen-Effros type on a given Banach space E a commutative C^*-algebra $Z(E)$ of $L(E, E)$ (called the centralizer) and discuss up to what extent a commutative C^*-subalgebra of $L(E, E)$ is necessarily a centralizer. Based on the results in this section we can assert that our theory of extremality is essentially commutative.

Section 5 is devoted to the spectral decomposition of compact operators in $Z(E)$. Our results cover classical Hilber-Schmidt spectral theorem and also all unconditional finite dimensional decompositions that arise in Banach space theory. Since the main ingredient in our proof is $Z(E)$ (rather than an explicit AE-order relation) one can reformulate all those results directly in terms of commutative von Neumann algebras (and

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also of Bade complete Boolean algebras of projections). However, due to Theorem 4.3 below, this is merely a question of gustibus.

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1. ALFSEN-EFFROS TYPE ORDER RELATIONS

Let E be a Banach space over the field $\mathbf{K}$ ($\mathbf{K}$ is $\mathbf{R}$ or $\mathbf{C}$).

1.1. *Definition.* An order relation $\ll$ on E is said to be of *Alfsen-Effros type* (abbreviated, $\ll$ is an *AE-order relation*), provided the following conditions are satisfied:

AE 1) $x \ll y$ implies $y - x \ll y$;

AE 2) $x \ll y$ implies $\alpha x \ll \alpha y$ for every $\alpha \in \mathbf{K}$;

AE 3) $0 \leqslant \alpha \leqslant \beta$ in $\mathbf{R}$ implies $\alpha x \ll \beta x$ for every $x \in E$;

AE 4) If $x_1 \ll y_1$, $x_2 \ll y_2$ and $y_1 \ll y_1 + y_2$ then $x_1 \ll x_1 + x_2$ and $x_1 + x_2 \ll y_1 + y_2$;

AE 5) $x + y \ll 2y$ implies $\|x\| \leqslant \|y\|$;

AE 6) $x_\alpha \ll y$ $(\alpha \in A)$ and $\|x_\alpha - x\| \to 0$ imply $x \ll y$.

Clearly, the definition above can be adapted in an evident manner for locally convex spaces with a specified system of seminorms. Also, one can rephrase conditions $AE1) - AE6)$ above in terms of codirection by letting

$x \| y$ (i.e., x and y are *codirectional*) if and only if $x \ll x + y$.

An AE — order relation is not compatible with the linear structure. In fact, $AE1)$ yields

$$0 \ll x \text{ for every } x \text{ in } E.$$

Examples:

a) The 1-dimensional Banach space $\mathbf{K}$ admits only one AE-order relation

$x \ll_C y$ if and only if $x = \alpha y$ for a suitable $\alpha \in [0, 1]$.

b) If H is a Hilbert space and $\mathscr{A}$ is a commutative von Neumann subalgebra of $L(H, H)$ then we may consider on H the following AE-order relation

$x \ll_{\mathscr{A}} y$ if and only if $x = Ay$ for a suitable $A \in \mathscr{A}$, $0 \leqslant A \leqslant I$. Here I denotes the identity of H. Details will be found in [N2].

c) On each regularly ordered Banach space E (in the sense of E. B. Davies [D]) we can consider the AE-order relation: $x \ll_o y$ if and only if every order interval $[u, v]$ of E containing 0 and y contains also x. See [N1] for details. For E a Banach lattice, $x \ll_o y$ is equivalent to $|y| = |x| + |y - x|$.

d) The following two examples make sense for every Banach space E and they were first considered by Alfsen and Effros [AE]

$$x \leqslant_L y \text{ if and only if } \| y \| = \| x \| + \| y - x \|$$

$x \leqslant_M y$ if and only if every closed ball containing 0 and y contains also x.

1.2 *Definition.* Given a Banach space E endowed with an AE-order relation $\leqslant$, we shall say that the norm of E is ($\leqslant$-) *order continuous* provided every $\leqslant$-decreasing net $(x_\alpha)_\alpha$ of elements of E is norm convergent.

If the norm of E is $\leqslant$-continuous then every $\leqslant$-increasing $\leqslant$-majorized net of elements of E is also norm convergent. In fact, if $(x_\alpha)_\alpha$ is such a net and $x_\alpha \leqslant y$ for every α, then the net $(y - x_\alpha)_\alpha$ is $\leqslant$-decreasing. The argument is as follows: $x_\alpha \leqslant x_\beta \leqslant y$ impilies $y - x_\beta \leqslant y$ and $x_\beta - x_\alpha \leqslant x_\beta \leqslant y$, so by AE4) above we infer that $y - x_\beta \leqslant (y - x_\beta) + (x_\beta - x_\alpha) = y - x_\alpha$.

For $u, v \in E$ with $u \leqslant v$ we define the ($\leqslant$-) *interval* of extremities u and v as the set

$$[u, v] = \{x ; x \in E, \ u \leqslant x \leqslant v\}.$$

Every interval $[u, v]$ is convex and norm closed. Use AE4) and AE6) above. An immediate consequence is the fact that every decreasing net $(x_\alpha)_\alpha$ of elements of a Banach space with an order continuous norm is norm convergent to $\leqslant$- inf x_α.

For E a Banach lattice and $\leqslant = \leqslant_o$, Definition 1.2 above agrees with the classical concept of order continuity as known in Banach lattice theory. See [LT2], 7, or [S], p. 92.

As was noticed in [AE], p. 107 the norm of every Banach space is $\leqslant_L$-continuous. Others examples come through the following result.

1.3 *Dini's Generalized Lemma* (See [N3]). *Suppose that* $(x_\alpha)_\alpha$ *is a* $\leqslant$*-decreasing net of elements of* E*, weakly converging to* x*. Then* $\| x_\alpha - x \| \to 0$.

1.4 Corollary (See [N3]). *Let* E *be a Banach space endowed with an* AE*-order relation* $\leqslant$ *such that all* $\leqslant$*-intervals* $[u, v]$ *of* E *are weakly compact. Then the norm of* E *is* $\leqslant$*-order continuous.*

By Corollary 1.4 we infer that the norm of every reflexive Banach space E is a order continuous whatever order relation $\leqslant$ we consider on it.

2. FACIAL CONES

In what follows E will denote a Banach space endowed with an AE-order relation $\leqslant$.

By a *cone* of E we shall mean every non-empty subset C of E such that $E = \bigcup_{\alpha \geqslant 0} \alpha C$; C is said to be *proper* (respectively *convex*) provided $C \cap (-C) = \{0\}$ (respectively $(C + C \subset C)$. By a ($\leqslant$-) *facial cone* of E

we shall mean every convex cone C of E that satisfies the following two conditions :

FC 1) $x \ll y$ and $y \in C$ imply $x \in C$;

FC 2) $x \parallel y$ for every x and y in C.

For $\ll \, = \, \ll_L$ we retrieve the classical notion of a facial cone. See [AE], page 106.

By *FC*2) and anti-symmetry of $\ll$ we infer that every facial cone is proper.

Each $x \in E$ belongs to a facial cone, e.g. to

$$C(x) = \{y \,;\, y \in E, \ \exists\ \alpha \geqslant 0, \ y \geqslant \alpha x\}$$

$C(x)$ is the smallest facial cone containing x and thus *the facial cone generated by* x.

The intersection of a facial cone of E with the unit sphere S of E will be called a *face* (of the unit ball K of E); in this setting the face generated by an $x \in S$ will be

$$\text{face}\ \{x\} = C(x) \cap S.$$

A facial cone may not be closed, e.g., see the case where $E = L^2[0,1]$ $\ll \, = \, \ll_o$ and $C = C(1)$; in this particular case $\bar{C} = E_+$. We, shall prove later that for finite dimensional Banach spaces all cones $C(x,$ are closed.

The facial cones allow us to develop an ideal theory that is in many respects comparable with that in Banach lattice theory. For, we need an observation, important also for itself.

To any convex proper cone C of a vector space E we can associate an ordering on E compatible with the linear structure

$$x \leqslant y \ (\text{mod}\ C) \text{ if and only if } y - x \in C.$$

2.2 Lemma. *If E is endowed with an Alfsen-Effros type order relation $\ll$ and x and y are two elements of E then the following assertions are equivalent*:

i) $x \ll y$;

ii) $0 \leqslant x \leqslant y$ (mod C) *for a suitable facial cone C containing y*;

iii) $0 \leqslant x \leqslant y$ (mod C) *for every facial cone C containing y*.

Proof. *i*) $\Rightarrow$ *iii*). If $x \ll y$ and C is a facial cone containing y then x, $y - x \in C$ because C is hereditary.

Clearly, *iii*) $\Rightarrow$ *ii*).

ii) $\Rightarrow$ *i*) . By hypotheses, x and $y - x$ are in C. By *FC*2), $x \parallel y - x$ and thus $x \ll y = x + (y - x)$. ∎

The principal ideal generated by an element x of E is defined as the set $E_x = \text{Span}\ C(x)$. The real part of E_x,

$$\mathcal{R}e\, E_x = C(x) - C(x)$$

will be endowed with the ordering associated to $C(x)$ and the norm

$$\|y\|_x = \inf\{\alpha\,;\, \alpha \geqslant 0, \ \exists\ u \ll \alpha x, \ v \ll \alpha x, \ y = u - v\}.$$

The fact that $\| y \|_x = 0$ implies $y = 0$ can be proved as follows: Let u_n, $v_n \in C(x)$ with $y = u_n - v_n$ and $u_n, v_n \ll x/n$ for every $n \in \mathbf{N}^*$ By $AE5$), $\| u_n \|, \| v_n \| \leqslant \| x \|/n$, so by letting $n \to \infty$ we conclude that $y = 0$.

2.3 Lemma. *For $y \in E_x$ and $\alpha \geqslant 0$ the following assertions are equivalent:*

i) $y = u - v$, *where* $u, v \ll \alpha x$;

ii) $-\alpha x \leqslant y \leqslant \alpha x \pmod{C(x)}$;

iii) $y + \alpha x \ll 2\alpha x$.

The proof is obvious.

From Lemma 2.3. and $AE5$) we infer that the canonical inclusion $i_x : \mathcal{Re}\, E_x \to E$ is continuous and $\| i_x \| \leqslant \| x \|$

2.4 Lemma. *$\mathcal{Re}$ E_x is an ordered Banach space with a strong order unit (that is x).*

Proof. We have to prove only the completeness of $\mathcal{Re}\, E_x$. For, let $(y_n)_n$ be a Cauchy sequence in $\mathcal{Re}\, E_x$. Since i_x is continuous, $(y_n)_n$ is also a Cauchy sequence in E and thus there exists a $y \in E$ such that $\| y_n - y \| \to 0$. On the other hand for each $\varepsilon > 0$ there exists an N such that

$$\| y_m - y_n \|_x < \varepsilon \quad \text{for every } m, n \geqslant N$$

i.e.,

$$-\varepsilon x \leqslant y_m - y_n \leqslant \varepsilon x \bmod C(x) \text{ for every } m, n \geqslant N.$$

Use Lemma 2.3. above. By letting $m \to \infty$ we infer that $-\varepsilon x \leqslant y - y_n \leqslant \varepsilon x \bmod C(x)$ for every $n \geqslant N$ which yields $y \in E_x$ and $\| y - y_n) \|_x \to 0$. ■

The following proposition combines classical results due to Kadison, Kakutani and Krein.

2.5 Proposition. *i) $\mathcal{Re}$ E_x is algebraic isometric and order isomorphic to a space $A(K, \mathbf{R})$ of all continuous affine real functions defined on the w' — compact convex set K of all states of $\mathcal{Re}\, E_x$.*

ii) Suppose in addition that

1) *either $\mathcal{Re}$ E_x is endowed with a bilinear multiplication for which x is an identity and*

$$y, z \in \mathcal{Re}\, E_x,\ y \geqslant 0 \text{ and } z \geqslant 0 \text{ imply } yz \geqslant 0; \text{ or,}$$

2) *$\mathcal{Re}$ E_x is a vector lattice with respect to the ordering mod $C(x)$.*

Then $\mathcal{Re}$ E_x is a commutative Banach algebra algebraic isometric and order isomorphic to a space $C(S, \mathbf{R})$, where S denotes the set of all pure states of $\mathcal{Re}\, E_x$.

Proof. For *i*), see [Kad2]; *ii* 1) follows from [Kad 1], while *ii* 2) needs the classical representation theorem of AM-spaces due to Kakutani and Krein. ■

3. EXTREME POINTS

Let E be a Banach space endowed with an AE-order relation $\leqslant$. We shall denote by K the unit ball of E and by S its unit sphere.

3.1 *Definition.* A norm 1 element x of E will be called ($\leqslant$-) *extremal* for K provided $C(x) = \mathbf{R}_+ \cdot x$.

In terms of faces, an extreme point is precisely a norm 1 vector x such that

$$\text{face } \{x\} = \{x\}.$$

Since K is the only subset of E whose extreme points are investigated, we shall denote by Ex E(or Ex$_\leqslant$ E) the subset of all extreme points of the unit ball of E. In order to avoid sub-scripts, we shall use notation like Ex$_L$ E in case $\leqslant = \leqslant_L$.

3.2 Lemma. *Let H be a Hilbert space and $\mathscr{A}$ the von Neumann subalgebra of $L(H, H)$ generated by a self-adjoint operator $A \in L(H, H)$. Then* Ex$_{\mathscr{A}}$ *H consists of all normalized eigenvectors of A.*

Proof. Let $v \in \text{Ex}_{\mathscr{A}} H$. Because $0 \leqslant A_-, A_+ \leqslant \|A\| \cdot I$ and A_-, $A_+ \in A$ it follows that

$$A_+ v, A_- v \leqslant_{\mathscr{A}} \|A\| \cdot v$$

and thus $Av = A_+ v - A_- v = \alpha v$ for a suitable $\alpha \in \mathbf{R}$.

Conversely, let $Av = \alpha v$ with $\|v\| = 1$ and $\alpha \in \mathbf{R}$. Then $f(A) \cdot v = f(\alpha) \cdot v$ for every $f \in C(\sigma(A))$ i.e., v is an eigenvector for every operator in the C^*-algebra $C^*\{A, I\}$ generated by A and I. Since $\mathscr{A}$ is the wo-closure of $C^*\{A, I\}$, then the same is true for every operator in $\mathscr{A}$. Consequently $x \leqslant_{\mathscr{A}} v$ implies $x = \lambda v$ for a suitable $\lambda \geqslant 0$ i.e., $v \in \text{Ex}_{\mathscr{A}} H$. ■

The notion of an extreme point is very closed to that of a discrete element.

3.3 *Definition.* By a ($\leqslant$-) *discrete element* of E we shall mean any element x of E such that

$$u, v \leqslant x \text{ and } C(u) \cap C(v) = \{0\} \text{ imply either } u \text{ or } v \text{ is null.}$$

Clearly, every element of Ex E is discrete. Conversely every normalized discrete element is also an extreme point. Before indicating the details, we shall notice without proof the particular case of Banach lattices :

3.4 Proposition. *Let E be a Banach lattice.*

i) An element x of E is $\leqslant_e$ -discrete if and only if it is an atom i.e., $u, v \leqslant |x|$ and $u \wedge v = 0$ imply u or v is null.

ii) Ex$_e$ *E coincides with the set of all normalized atoms of E.*

Consequently, in the real case,

$\mathrm{Ex}_L\ c_0 = \emptyset$ $\mathrm{Ex}_o\ c_0 = \{\pm(\delta_{mn})_m\,;\ n \in \mathbf{N}^*\}$

$\mathrm{Ex}_L\ C[0,1] = \{\pm 1\}$ $\mathrm{Ex}_o\ C[0,1] = \emptyset$

$\mathrm{Ex}_L\ L^2[0,1] = \{x\,;\ x \in L^2[0,1],\ \|x\| = 1\}$

$\mathrm{Ex}_o\ L^2[0,1] = \emptyset$.

We shall prove that in general every normalized discrete element is an extreme point. Our argument is essentially finite dimensional and depends upon an analogue of the orthogonal decomposition.

3.5 Lemma. *Let E be a finite dimensional Banach space endowed with an AE-order relation $\ll$ and let $x \in E$, $x \neq 0$.*

Then the cone $C(x)$ is closed and for every $e \in E$ there exist elements u and v in $[0, e]$ such that

$$e = u + v$$

$u \in C(x)$ *and* $C(v) \cap C(x) = \{0\}$.

Proof. We shall show first that the cone $C(x)$ is closed. For, let $(y_n)_n$ be a sequence of elements of $C(x)$ such that $\|y_n - y\| \to 0$ in E. Then

$$y_n \ll \|y_n\|_x \cdot x \quad \text{for every } n \in \mathbf{N}^*.$$

Since $\dim E < \infty$, the canonical inclusion $i_x : E_x \to E$ is an isomorphism into and thus $y \in E_x$ and $\|y_n - y\|_x \to 0$. Put $M = \sup\|y_n\|_x$. By AE 3) and AE 6) above we infer that $y \ll Mx$ i.e., $y \in C(x)$.

As concerns the decomposition part, consider the set $A_e = \{z\,;\ z \in C(x),\ z \ll e\}$; $0 \in A_e$ and A_e is inductively ordered by $\ll$. In fact, the order interval $[0, e]$ is compact and thus every increasing net of elements of $[0, e]$ is norm convergent to its l.u.b. By Zorn's Lemma, A_e must contain at least one maximal element, say u. It remains to prove that $v = e - u$ satisfies $C(v) \cap C(x) = \{0\}$. In fact, if the contrary is true then it would exist a $z \in C(x)$ such that $z \neq 0$ and $z \ll v$. Since

$$z \ll e - u$$

$$u \ll u$$

$$e - u \parallel u,$$

by AE 4) it follows that $z + u \ll e$ and $u \ll z + u$. Since $C(x)$ is convex, $z + u \in C(x)$ and this fact contradicts the maximality of u. Consequently $C(v) \cap C(x) = \{0\}$. ∎

3.6 Lemma. *Let E be a finite dimensional Banach space endowed with an AE-order relation $\ll$. Then* Ex E *consists precisely of all normalized discrete elements of E.*

Proof. Suppose that e is a normalized discrete element of E and $e \notin \operatorname{Ex} E$. Then there exists an $x \in E$ such that $x \leqslant e$ and $x \notin \mathbf{R}_+ \cdot e$. Put

$$\alpha = \sup \{\lambda ; \ \lambda e \leqslant x\}.$$

Then $x - \alpha e \neq 0$ and $C(x - \alpha e) \cap \mathbf{R}_+ \cdot e = \{0\}$. In fact, if $\mu e \leqslant \leqslant \lambda(x - \alpha e)$ with $\mu, \lambda \in \mathbf{R}_+^*$ then $(\mu + \lambda\alpha)e \leqslant \lambda x$ i.e., $(\mu/\lambda + \alpha)\, e \leqslant x$, in contradiction with the definition of α. By Lemma 3.5, e admits a decomposition

$$e = u + v$$

with $u \in C(x - \alpha e)$ and $C(v) \cap C(x - \alpha e) = \{0\}$. We shall prove that both u and v are different to 0, which will contradict the fact that e is discrete.

If $v = 0$, then $e = u \in C(x - \alpha e)$. Or, $C(x - \alpha e) \cap \mathbf{R}_+ \cdot e = \{0\}$.

If $u = 0$ then $e = v$ and thus $C(e) \cap C(x - \alpha e) = \{0\}$; Or, $x - \alpha e \leqslant \leqslant x \leqslant e$, so that $C(x - \alpha e) \subset C(e)$.

Consequently e is an extreme point of E.

The fact that every extreme point is also a discrete element was already remarked. ■

3.7 Theorem. *Let E be a Banach space endowed with an AE-order relation $\leqslant$. Then* Ex E *consists precisely of all normalized discrete elements of E.*

Proof. We have only to prove that every normalized discrete element e of E is also an extreme point i.e.,

$$C(e) = \mathbf{R}_+ \cdot e.$$

For, notice that $\leqslant$ induces on each finite dimensional subspace F of E an AE-order relation $\leqslant_F$ given by

$$x \leqslant_F y \text{ if and only if } x \text{ and } y \text{ belong to } F$$

$$\text{and } x \leqslant y \text{ in } E.$$

By Lemma 3.6, $C(e) \cap F = \mathbf{R}_+ \cdot e$. for every finite dimensional subspace F which contains e and thus $C(e)$ is indeed $\mathbf{R}_+ \cdot e$ ■.

It is perhaps of some interest to outline the connection with the notion of an extremal ray : Let C be a facial cone and $x \in C$, $x \neq 0$. Then x is discrete if and only if the ray $\mathbf{R}_+ \cdot x$ generated by x is extremal i.e.,

$$x = tu + (1-t)v \text{ with } u, v \in C \text{ and } 0 < t < 1 \text{ implies } u, v \in \mathbf{R}_+ \cdot x.$$

As an immediate consequence we infer that every extreme point $e \in \operatorname{Ex} E$ is also an extreme point in the classical sense for $C \cap K$ where C is any facial cone containing x. The converse does not work e.g., consider the Euclidian plane $\mathbf{R}^2$ endowed with the AE-order relation $\leqslant_o$.

Once the notion of an ($\ll$-) extreme point precised it is important to know up to what extent the given Banach space E is generated by Ex E. A remark above suggests to apply Choquet's extremal decomposition of well capped cones. See [Ph], ch.11. Unfortunately, that elegant approach does not cover all important particular cases where a nice decomposition of the given space is possible. We shall prove instead an analogue of Hilber-Schmidt Theorem which brings together several types of finite dimensional decompositions including the orthogonal and the lattice one. See Theorem 5.1 below. In turn, our approach seems to leave out classical results like Krein-Milman Theorem.

The proof of Theorem 5.1 depends heavily on the finite dimensional case treated below by using the decomposition method described in Lemma 3.5; that case is covered also by Choquet's theory. The remainder is a generalization of the notion of a self-adjoint operator and will be presented in the next section.

3.8 LEMMA. *If C_1 and C_2 are facial cones such that $C_1 \subset C_2$ and $C_1 \neq C_2$ then $C_1 \subset \operatorname{Fr} C_2$.*

Proof. Suppose that the contrary is true. Then exist an $x \in C_2$ and an $r > 0$ such that $B_r(x) \cap C_2 \subset C_1$. If $z \in C_2 \setminus C_1$ and $\|z\| < r$ then $x + z \in B_r(x) \cap C_2 \subset C_1$. Since x, $z \in C_2$ we have $z \ll x + z$. Since C_1 is hereditary, the last implies that $z \in C_1$, a contradiction. ∎

3.9 THEOREM. *Suppose that* dim $E = n$ *and E is endowed with an AE-order relation $\ll$. Then for each $x \in E$ there exist scalars $\alpha_1, \ldots, \alpha_n \in [0, \|x\|]$ and $\ll$- extreme points $e_1, \ldots, e_n \in C(x)$ such that*

$$x = \alpha_1 e_1 + \ldots + \alpha_n e_n.$$

Proof. The assertion is clear for x an extreme point.

Suppose that $x \notin \operatorname{Ex} E$ and $\|x\| = 1$. We shall prove first that there exist discrete elements f such that $f \ll x$ and $f \neq 0$. In fact, by Lemma 3.6 above there exist elements $u, v \in 0[0, x] \setminus \{0\}$ such that $C(u) \cap C(v) = \{0\}$. By Lemma 3.8, $C(u) \subset \operatorname{Fr} C(x)$ and thus $\dim C(u) < \dim C(x) \leqslant n$. Consequently in at most n steps we are led to a discrete element f_1 with $f_1 \ll x$ and $f_1 \neq 0$. Put $e_1 = f_1/\|f_1\|$ and

$$\alpha_1 = \sup\{\lambda\,;\ \lambda e_1 \ll x\}.$$

Then $e_1 \in \operatorname{Ex} E$ and $C(x - \alpha_1 e_1) \cap C(e_1) = C(x - \alpha_1 e_1) \cap \mathbf{R}_+ \cdot e = \{0\}$. If $x - \alpha_1 e_1$ is not discrete the process described above should be continued with $x - \alpha_1 e_1$ instead of x. ∎

For $\ll = \ll_L$, Theorem 3.9 shows that every x in the unit sphere of E is a convex combination of extreme points. In fact, if

$$x = \alpha_1 e_1 + \ldots + \alpha_n e_n$$

with $e_1, \ldots, e_n \in C(x)$ and $\alpha_1, \ldots, \alpha_n \geqslant 0$ then by FC 2),

$$1 = \|x\| = \|\alpha_1 e_1\| + \ldots + \|\alpha_n e_n\| = \alpha_1 + \ldots + \alpha_n.$$

Since $e \in \operatorname{Ex} E \cap C(x)$ if and only if $-e \in \operatorname{Ex} E \cap C(-x)$. Theorem 3.9. includes the classical result due to Caratheodory that states that each point in the unit ball of an n-dimensional Banach space is a convex combination of at most $n+1$ extreme points. (In turn, Theorem 3.9 is an easy consequence of Caratheodory's result).

It includes also the following result due to Yudin: Every finite dimensional Banach lattice has a basis formed by atoms (and thus it is algebraic topologic and lattice isomorphic to a space $\mathbf{R}^n$).

From Theorem 3.9 we can deduce easily the fact that given an $n\times n$-dimensional self-adjoint matrix A there exists an orthonormal basis of $\mathbf{C}^n$ formed by eigenvectors of A.

4. THE CENTRALIZER

As usualy, E will denote a Banach space endowed with an AE-order relation $\leqslant$. We can associate to $\leqslant$ an AE-order relation on $L(E, E)$ (endowed with the family of all semi-norms $A \to \|Ax\|$, $x \in E$) by letting

$$A \leqslant B \text{ if and only if } Ax \leqslant Bx \text{ for every } x.$$

The centralizer associated to $\leqslant$ is defined as the linear span $Z(E)$ of the facial cone

$$Z(E)_+ = \{A\,;\, A \in L(E,\ E),\ \exists\ \alpha \geqslant 0, A \leqslant \alpha I\}.$$

Notations like $Z_{\leqslant}(E)$ and $Z_L(E)$ are destinated to precise that the AE-order relation under study is $\leqslant$, respectively $\leqslant_L$.

As in case $\leqslant = \leqslant_M$, first studied in [AE], one can prove that $\mathcal{Re}\, Z(E)$ is an ordered Banach algebra endowed with the cone $Z(E)_+$ and the norm $\|\ \|_I$ associated to the strong order unit I.

4.1 THEOREM. *$\mathcal{Re}\ Z(E)$ is algebraic isometric and order isomorphic to an ordered Banach algebra $C(S, \mathbf{R})$, where S denotes the Gelfand spectrum of $\mathcal{Re}\ Z(E)$.*

Moreover, $\|\ \|$ and the operatorial norm coincide on $\mathrm{Re}Z(E)$.

Proof. By Proposition 2.5 *ii*), it suffices to prove that $\|\ \|$ and $\|\ \|_I$ coincide on $\mathcal{Re}\, Z(E)$. For, let $A \in \mathcal{Re}\ Z(E) = C(S, \mathbf{R})$ and suppose that $\alpha = \|A\|_I = \inf\{\lambda > 0\,;\ A \leqslant \lambda I\} > 0$. Then for each $\varepsilon \in (0, \alpha)$ there exists a $U \in \mathcal{Re}\ Z(E)$ such that $0 \leqslant U \leqslant I$, $U \neq 0$, an $AU \geqslant (\alpha-\varepsilon)\ U \geqslant 0$ i.e., $(\alpha-\varepsilon)U \leqslant AU$. Since $U \neq 0$, there exists an $x \in E$ such that $y = Ux \neq 0$. Then $(\alpha-\varepsilon)\ y \leqslant Ay$, so by $AE\,5)$, $(\alpha-\varepsilon)\ \|y\| \leqslant \|Ay\|$. Consequently $\|A\| \geqslant \alpha-\varepsilon$. The inequality $\|A\| \leqslant \|A\|_I$ also follows from $AE\,5)$. ∎

By Theorem 4.1, $\mathrm{Re}Z(E)$ is formed by self-adjoint operators provided E is a complex Banach space. Recall that a bounded linear operator A acting on a complex Banach space E is said to be *self-adjoint* (or hermitian) provided $\|e^{itA}\| = 1$ for every $t \in \mathbf{R}$. That motivates the terminology of ($\leqslant$-) self-adjoint operators for the elements of $\mathcal{Re}\ Z(E)$.

By Theorem 4.1, the idempotents of $Z_{\ll}(E)$ (the so called *$\ll$-Cunningham projections*) are precisely those projections $P \in L(E, E)$ such that

$$Px \ll x \text{ for every } x \in E.$$

See [N2—N3] for details and comments.

The $\ll$-Cunnigham projections of E constitute a Boolean algebra of projections $\mathbf{P}_{\ll}(E)$ (denoted also by $\mathbf{P}(E)$ when $\ll$ is understood) if we put

$$P \vee Q = P + Q - PQ$$

$$P \wedge Q = PQ$$

$$P^c = I - P.$$

4.2 Lemma. *Suppose that the norm of E is $\ll$-continuous. Then*

i) $Z(E) = \text{Span } \mathbf{P}(E)$;

ii) $\mathbf{P}(E)$ *is a Bade-complete Boolean algebra of projections.*

Proof. i) Since the norm of E is $\ll$-continuous, $Z(E)$ is an order complete $C(S)$-space; for, combine Theorem 4.1 with the fact that every increasing majorized net $(A_\alpha)_\alpha$ in $Z(E)$ is point-wise convergent (necessarily to its l.u.b.). Then S has a basis of open-closed sets and thus by a remark above $\mathbf{P}(E)$ separates the points of S. The proof ends by applying Stone-Weierstrass Approximation Theorem.

ii) We have to show that for each net $(P_\alpha)_\alpha$ in $\mathbf{P}(E)$, $\vee P_\alpha$ and $\wedge P_\alpha$ exist in $\mathbf{P}(E)$ and moreover

$$(\vee P_\alpha)(E) = \overline{\text{Span}} \cup P_\alpha(E)$$

$$(\wedge P_\alpha) E = \cap P_\alpha(E).$$

For, it suffices to consider increasing nets $(P_\alpha)_\alpha$. Then $P(x) = \lim P_\alpha(x)$ exists in the norm topology of E for each x, $P^2 = P$ and by $AE\,6)$, $P_\alpha \ll P \ll I$ for each α. Consequently $P = \vee P_\alpha$ in $\mathbf{P}(E)$ and $P(E) = \overline{\text{Span}} \cup P_\alpha(E)$. ∎

The result of Lemma 4.2 can be considerably improved and we shall state here without proof the existence of a one-to-one correspondence between AE-order relations and commutative algebras. That motivates the attribute of *commutative* for our theory of extremality. The details are to be found in [N2].

4.3 Theorem. *i) Suppose that the norm of E is order continuous. Then $Z(E)$ is a commutative von Neumann algebra and the canonical inclusion $i : Z(E) \to L(E, E)$ maps w^*-convergent nets into wo-convergent nets.*

ii) Suppose that $\mathscr{A}$ is a commutative von Neumann algebra included by $L(E, E)$ such that the canonical inclusion $i : \mathscr{A} \to L(E, E)$ is a morphism of unital Banach algebras mapping w^*-convergent nets into wo-convergent nets.

Then there exists an AE-order relation $\leqslant_{\mathscr{A}}$ *on* E *such that the norm of* E *is* $\leqslant_{\mathscr{A}}$*-continuous and* $Z_{\mathscr{A}}(E) = \mathscr{A}$.

Precisely,

$$x \leqslant_{\mathscr{A}} y \text{ if and only if } x = Ay \text{ for a suitable } A \in \mathscr{A}, 0 \leqslant A \leqslant I$$

The order relation $\leqslant_{\mathscr{A}}$ is maximal among those having $\mathscr{A}$ as a centralizer. In fact, if $Z_{\leqslant}(E) = \mathscr{A}$ then

$$x \leqslant_{\mathscr{A}} y \text{ implies } x \leqslant y.$$

Few comments concerning the connection between ExE and $Z(E)$ are in order. Every extreme point $e \in \text{Ex } E$ is also an eigenvector to every $A \in Z(E)$. Consequently, if there exists an $A \in Z(E)$ with no eigenvector then Ex $E = \emptyset$. On the other hand, it is possible to find eigenvectors for $Z(E)$ which are not discrete elements e.g., consider the case where E is the Euclidian plane and $\leqslant = \leqslant_{o}$; in that case $Z(E) = \mathbf{R} \cdot I$ while card Ex E $= 4$. A nice exception constitutes the order relation $\leqslant_{\mathscr{A}}$.

5. THE SPECTRAL DECOMPOSITION OF $\leqslant$-SELF-ADJOINT OPERATORS

The classical Hilber-Schmidt Theorem states that every self-adjoint compact operator A acting on a Hilbert space H diagonalizes with respect to a suitable orthonormal basis formed by eigenvectors i.e., A admits a representation of the form

$$(HS) \qquad Ax = \sum_n \lambda_n \langle x, v_n \rangle v_n.$$

If $\mathscr{A}$ denotes the von Neumann subalgebra of $L(H, H)$ generated by A then the norm of H is $\leqslant_{\mathscr{A}}$-continuous (H is reflexive and thus Proposition 1.3 applies). Each finite rank projection

$$P_n(x) = \sum_{\lambda_k = \lambda_n} \langle x, v_k \rangle v_k.$$

belongs to $\mathscr{A}$ and so they are $\leqslant_{\mathscr{A}}$-Cunnigham projections. Since (HS) yields

$$A = \sum \lambda_n P_n$$

in the norm topology of $L(H, H)$, it follows that $A \in \mathscr{R}e\, Z_{\mathscr{A}}(H)$.

It is remarkable that the results above remain valid in the general setting.

5.1 Hilbert-Schmidt-Generalized Theorem. *i*) *Let* E *be a Banach space endowed with an AE-order relation* $\leqslant$ *such that the norm of* E *is* $\leqslant$*-continuous. Then for every compact operator* $A \in \mathscr{R}eZ(E)$ *there exist a real sequence* $(\alpha_n)_n \in c_0$ *and a sequence* $(P_n)_n$ *of mutually disjoint finite rank Cunningham projections on* E *such that*

$$Ax = \sum_n \alpha_n P_n(x) \text{ for every } x \in E.$$

ii) Conversely, every operator $A \in L(E, E)$ which admits such a representation is compact and belongs to $\mathscr{R}e\, Z(E)$.

Proof. i) The non-trivial case is when $A \neq 0$. Notice first that $U \leqslant V$ and V compact imply U is compact too. In fact, for each sequence $(x_n)_n$ of elements of E we have $U(x_m - x_n) \leqslant V(x_m - x_n)$ and thus $\|Ux_m - Ux_n\| \leqslant \leqslant \|Vx_m - Vx_n\|$ for all $m, n \in \mathbf{N}^*$.

Since $A_+, -A_-, \leqslant A$ we may restrict ourselves to the case where $A \in Z(E)_+$.

Since $Z(E)$ is order complete, A is the uniform limit of sums of the form $\sum_{i=1}^{n} \alpha_i P_i$ with $0 \leqslant \sum_{i=1}^{n} \alpha_i P_i \leqslant A$ and $\alpha_i \geqslant 0$ for every i. Then there exist scalars $\lambda > 0$ and finite rank Cunnigham projections $Q \neq 0$ such that $\lambda Q \leqslant A$. The image of Q is an invariant subspace of A and thus the argument above shows that $AQ = \sum \lambda_n Q_n$ for suitable $Q_n \in \mathbf{P}(E)$, $Q_n \neq 0$ and $\lambda_n > 0$. Consequently there exist scalars $\alpha > 0$ and finite rank Cunningham projections $P \neq 0$ such that $AP = \alpha P$; clearly, $\alpha \in \sigma(A) \setminus \{0\}$ and we can choose P to be maximal with this property. Since A is compact, card $\sigma(A) \leqslant \aleph_0$ and thus the set $(\alpha_n, P_n)_n$ of all such pairs is at most countable. We shall show that

$$A = \sum \alpha_n P_n$$

is the desired decomposition. In fact, the projections P_n are mutually disjoint and $\sum \alpha_n P_n = \sup \alpha_n P_n \leqslant A$ because E has order continuous norm. Put $B = A - \Sigma \cdot \alpha_n P_n$. Then $B = A(I - Q)$ where $Q = \bigvee P_n$; B is compact and belongs to $Z(E)_+$. To end the proof of $i)$ we must show that $B = 0$. For, use the argument above in case $B \neq 0$ to contradict the maximality of the P_n's.

ii) It suffices to show that if A admits a representation

$$Ax = \sum_n \alpha_n P_n(x)$$

with $(\alpha_n)_n \in (\mathscr{C}_0)_+$ then $A = \sum \alpha_n P_n$ in the norm topology. Or, by AE 3) and AE 4), above,

$$\left\| Ax - \sum_{k=1}^{n} \alpha_k P_k x \right\| = \| \sum_{k>n} = \alpha_k P_k x \| \leqslant (\sup_{k>n} \alpha_k) \cdot \| x \|$$

for every $n \in \mathbf{N}^*$ and $x \in E$. ∎

Notice that under the hypotheses of Theorem 5.1 i) if $A \in \mathscr{R}e\, Z(E)$ admits a representation

$$A = \sum_n \alpha_n P_n$$

with all $P_n \neq 0$ then $\sigma(A) \setminus \{0\} = \{\alpha_n ; \alpha_n \neq 0\}$ and

$$\| A \| = \sup | \alpha_n | = r_{\mathrm{sp}}(A).$$

By combining Theorem 5.1 with Theorem 3.9 above we obtain the following :

5.2 Corollary. *Let A be as in Theorem 5.1. Then.*

$$\overline{\operatorname{Im} A} = \overline{\operatorname{Span}}\,[(\operatorname{Im} A) \cap (\operatorname{Ex} E)]$$

In other words, $\overline{\operatorname{Im} A}$ is generated by its extreme points.

The following result arised during conversations with M. Joiţa.

5.3 Theorem. *Let E be a Banach space endowed with an AE-order relation $\leqslant$ such that the norm of E is $\leqslant$-continuous. Then an operator $A \in \mathcal{R}e\, Z(E)$ is nuclear if and only if it admits a representation*

$$Ax = \sum_n \lambda_n P_n(x),\ x \in E$$

with $$(\lambda_n)_n \in c_0,\ (P_n)_n \in \mathbf{P}(E),\ P_m \wedge P_n = 0 \text{ for } m \neq n$$

and

$$\sum_n |\lambda_n| \cdot \dim\ P_n(E) < \infty.$$

The classical Hilbert-Schmidt Theorem covers all orthonormal expansions because every orthonormal basis $(e_n)_n$ of a Hilbert space H can be regarded as a fundamental system of eigenvectors of a self-adjoint compact operator $A \in L\,(H,H)$.

It is worthwhile to precise the type of decomposition Theorem 5.1 yields. For, we need the following definition generalizing the notion of an unconditional basis : By an *unconditional decomposition* of a Banach space E we mean any sequence $(E_n)_n$ of closed subspaces of E such that every $x \in E$ has an unique representation of the form

$$x = \sum_{n=1}^{\infty} x_n \text{ with } x_n \in E_n \text{ for each } n,$$

the series being unconditionally convergent ; if in addition $\dim E_n < \infty$ for all n, $(E_n)_n$ is said to be *an unconditionally finite dimensional decomposition* (*UFDD*) of E. Under the above notation, the *lattice constant* $\gamma_L((E_n)_n$ of an unconditional decomposition $(E_n)_n$ is defined as the smallest $C \geqslant 0$ such that

$$\left\| \sum_{k=1}^{n} \alpha_k x_k \right\| \leqslant C \left\| \sum_{k=1}^{n} \beta_k x_k \right\|$$

for all scalars $|\alpha_k| \leqslant |\beta_k|$, $k \in \{1, \ldots, n\}$, $n \in \mathbf{N}^*$.

Even nice spaces like $C\,[0,1]$ and $L^1[0,1]$ fail to have an *UFDD*. The best can be said at present about a space having an *UFDD* is that it is isomorphic to a subspace of a space with an unconditional basis. See [LT 1], p. 51.

A more careful look at the proof of Theorem 5.1 yields the following

5.4 Proposition. *Let A be as in Theorem 5.1. Then*

$$E = \overline{\operatorname{Im} A} \oplus \operatorname{Ker} A$$

is an unconditional decomposition of lattice constant 1 *and* $\overline{\mathrm{Im}\, A}$ *admits an UFDD of lattice constant* 1.

We shall show that via a renorming process all complemented subspaces having an *UFDD* arise in this way.

Let X be a closed subspace of a Banach space E and suppose that X admits an *UFDD* $(X_n)_n$ and there exists a projection P of E onto X. Consider also the canonical projections $P_n : x \to x_n$ of X onto X_n. Then E can be renormed by

$$|||x||| = \sup \{ \|\sum \alpha_n P_n P x\| + \|x - Px\| ; \ |\alpha_n| \leqslant 1, \ n \in \mathbf{N}^* \}.$$

On $(E, ||| \ |||)$ consider the AE-order relation $\ll$ given by

$$x \ll y \text{ if and only if } P_n P x = \lambda_n P_n P y \text{ and}$$
$$(I - P)\, x = \lambda\, (I - P)\, y \text{ for suitable scalars}$$
$$\lambda_n, \ \lambda \in [0, 1] \text{ and } n \in \mathbf{N}^*.$$

Under the above renorming, the *UFDD* decomposition $(X_n)_n$ of X coincides with that produced (via Proposition 5.3) by the operator

$$A = \sum_{n=1}^{\infty} \frac{1}{n^2} P_n P.$$

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Department of Mathematics
University of Craiova
Craiova, Romania

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